

Dendroidal weak 2-categories

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We discuss the dendroidal notion of weak higher categories introduced by Moerdijk and Weiss in [11] and we prove that dendroidal weak 2-categories are equivalent to bicategories.

55U10, 55P48, 55U40, 18G30, 18D50, 18D10;

1 Introduction

Weakened notions of categories are central in many branches of mathematics. One would often like to form certain “categories” where the composition of arrows is not strictly associative, but only up to some coherent higher cells that should be part of the structure. Important examples of such structures in the literature are homotopy n -types. Roughly speaking, a homotopy n -type in topological spaces is the equivalence class of a space X such that all the homotopy groups $\pi_k(X)$ are trivial when $k > n$. These classes are taken with respect to weak homotopy. Describing algebraic models for homotopy n -types is a classical problem in algebraic topology. For $n = 2, 3$ the first such models were given by Whitehead and Mac Lane [15, 10]. Following the influence of Grothendieck and R. Brown, who emphasized that groupoids should provide the natural framework for homotopy types, higher categorical algebraic models of homotopy 3-types were given and studied by Leroy [9], Joyal and Tierney [7], Berger [2].

If we work out the topological intuitions coming from the interplay between spaces, maps and homotopies between maps, we arrive to the abstract notion of bicategories, first defined by Bénabou in [1]. Bicategories are structures that consist of 0-cells (objects), 1-cells (arrows) and 2-cells; 1-cells are composable but not strictly associatively, the failure of this is measured by some natural 2-cells. We can iterate the process to arrive to a definition of tricategories and so on, the n -th step of this process would give us a notion of weak n -categories. The problem we encounter is that each step we take for defining a one-level higher notion increases radically the complexity of the necessary coherence conditions between the higher dimensional cells. These steps also diminish the intuition on the nature of the coherence axioms. As a result, there exist a

plethora of different definitions of weak n -categories in the literature. Comparing the different notions of weak n -categories is one of the main problems in higher category theory. One of the issues can be formulated as follows. Intuitively, the right place to compare two notions of weak n -categories would be inside a weak $n + 1$ -category, but how do we decide which notion of weak $n + 1$ -categories to use for this comparison?

To deal with these problems, one can consider stricter- or non-iterative approaches to define weak n -categories. The idea is that the resulting stricter notions should be enough to deal with the applications on the one hand, and the slogan is that “*weak n -categories are strictifiable up to some extent*” on the other. Examples of this approach include Baez and Dolan’s notion of (∞, n) -categories, Tamsamani categories, etc. We would like to mention explicitly one such example, originating in the observation that the category of categories embeds to simplicial sets, via the nerve functor:

$$N: \mathcal{Cat} \longrightarrow sSets.$$

Certain simplicial sets which are not in the image of the nerve functor behave much like categories, except that “composition” of arrows is well defined only up to some higher degree terms in that simplicial set. A. Joyal in [6] called these simplicial sets quasi-categories, although the notion was already introduced by Boardman and Vogt under the name of restricted Kan complex in [4]. A quasi-category is an $(\infty, 1)$ -category in Baez and Dolan’s sense, i.e. all the degree 2- or higher cells are invertible. An important fact about quasi-categories is that they are exactly the fibrant objects in the Joyal model structure for the category of simplicial sets.

The starting point of our investigation is that, there exist a generalisation of simplicial sets, that is suitable to study operads in the context of homotopy theory. On the one hand, operads (or rather coloured operads) can be viewed as generalisations of categories, where we consider arrows that can have multiple inputs as opposed to one. It is natural to ask whether there exists a presheaf category that extends the category of operads in the same way as simplicial sets extend the category of categories via the nerve functor. The question was studied in the papers of Moerdijk and Weiss [11, 12]: the category of dendroidal sets satisfies the requirements and fits in a commutative diagram of categories

$$\begin{array}{ccc} \mathcal{Cat} & \xrightarrow{N} & sSets \\ \downarrow & & \downarrow \\ \mathcal{Op} & \xrightarrow{N_d} & dSets \end{array}$$

Since dendroidal sets are an extension of simplicial sets, suitable for studying the homotopy theory of operads, the theory of dendroidal sets inherits a lot of questions

from the theory of simplicial sets. For example, this extension allows us to consider quasi-operads in the category of dendroidal sets, i.e. analogs of Joyal's quasi-categories. One can then ask whether the Joyal model structure on the category of simplicial sets extends to that of dendroidal sets in such a way, that the fibrant objects of this model category are exactly the quasi-operads. Cisinski and Moerdijk in [5] gave a positive answer to this question. One nice feature of dendroidal sets, observed in [11], is that they contribute to the theory of higher categories with a new compact definition of weak n -categories.

The aim of this paper is to study the dendroidal definition of weak n -categories mentioned above in low degree. We restrict ourselves to the cases $n = 1$ and $n = 2$. In the case of degree 2, the corresponding classical notion is bicategories. We prove that dendroidal weak 2-categories are equivalent to bicategories (even more, they are isomorphic as categories).

Our paper is organised as follows:

In Section 2 we introduce dendroidal sets, with emphasis on the necessary notions and terminology that will be used in the next Sections. The symmetric monoidal structure on the category of dendroidal sets that makes the definition of dendroidal weak n -categories possible is induced by the dendroidal nerve functor and the Boardman-Vogt tensor product of coloured operads. The dendroidal Grothendieck construction gives us a way to systematically glue together dendroidal sets, and is an important ingredient that will allow us to consider later dendroidal weak n -categories with any set of objects. The weakening of the higher dimensional cells in the dendroidal setting is done with the homotopy coherent dendroidal nerve functor, that is also introduced in this Section.

In Section 3 we define dendroidal weak n -categories and prove that they are 3-coskeletal for every n . This is an important property of dendroidal weak n -categories, since it implies that degree 0, 1 and 2 completely determines dendroidal weak n -categories.

In Section 4 first we describe dendroidal weak 1-categories. The iterative definition of dendroidal weak n -categories makes it then possible to discuss dendroidal weak 2-categories. We prove that the quasi-category of dendroidal weak 2-categories (denoted by $i^*(wCat^2)$) has equivalent homotopy category to the category of bicategories:

Theorem 4.5 *The category of unbiased bicategories $ubiCtg$ and $ho(i^*(wCat^2))$ are isomorphic. Hence the category of classical bicategories is equivalent to the category of dendroidal weak 2-categories.*

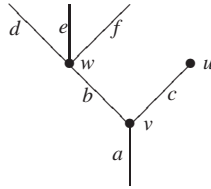
The definition and basic properties of classical and unbiased bicategories are recalled in the Appendix.

2 Dendroidal sets

2.1 Terminology and basic facts about dendroidal sets

Dendroidal sets generalise simplicial sets in a suitable way for studying the homotopy theory of (coloured) operads and their algebras. They were introduced in the papers of I. Moerdijk and I. Weiss [11, 12]. The idea behind the notion of dendroidal sets is that in the same way as simplicial sets help us understanding categories via the nerve functor, there should be an analogous notion for studying coloured operads as a generalisation of categories. Our goal here is to write a self-contained introduction to dendroidal sets, including all the terminology necessary for the next Sections.

Let us start with the notion of trees. A *tree* is a finite non-planar contractible graph with a distinguished leaf called *root*. A tree thus has many planar representatives, when we draw a picture of it we actually pick one. We will use the following terminology on trees: Cor_n denotes the n -corolla, i.e. the tree with one vertex and $n + 1$ leaves (one of these leaves is the root), $\text{Vert}(T)$ denotes the set of vertices of the tree T , $\text{Edg}(T)$ denotes the set of edges of the tree T and $\text{InEdg}(T)$ denotes the set of internal edges of the tree T . We will say that a vertex $v \in \text{Vert}(T)$ of a tree is an *outer vertex* if v is adjacent to at most one inner edge of T . For example, on the following picture of a (planar representative of a) tree T we have $\text{Vert}(T) = \{u, v, w\}$, $\text{Edg}(T) = \{a, b, c, d, e, f\}$, $\text{InEdg}(T) = \{c, b\}$. The vertices u and w are outer vertices.



We will make frequent use of symmetric coloured operads (both in *Sets* and enriched in a monoidal category \mathcal{E}) in the sense of [8] and we will refer to them as operads from now on. Recall that if P is an operad in *Sets*, then it comes equipped with a set of objects or colours $ob(P)$ and for each ordered sequence of objects $\sigma = (e_1, \dots, e_n; e)$, a set of operations $P(e_1, \dots, e_n; e) = P(\sigma)$. We will use the \circ_i -definition for the composition of operations, i.e. if σ is an ordered sequence or a *signature* as before and $\rho = (f_1, \dots, f_m; e_i)$ for a fixed $1 \leq i \leq n$ then

$$\sigma \circ_i \rho = (e_1, \dots, e_{i-1}, f_1, \dots, f_m, e_{i+1}, \dots, e_n; e)$$

and there is a given composition map

$$\circ_i: P(\sigma) \times P(\rho) \longrightarrow P(\sigma \circ_i \rho).$$

The category of operads in *Sets* will be denoted by \mathcal{Op} , and the category of planar- or non symmetric operads in *Sets* by \mathcal{Op}^π . Sometimes it will be useful to construct operads from planar ones, via the free symmetrization functor $\text{Symm}: \mathcal{Op}^\pi \rightarrow \mathcal{Op}$, the left adjoint to the forgetful functor $U: \mathcal{Op} \rightarrow \mathcal{Op}^\pi$. This feature already appears in the definition of dendroidal sets.

The category Ω^π consists of planar trees as objects and planar operad maps as arrows. To be more precise, any planar tree T gives rise to a planar operad $\Omega^\pi(T)$. The objects of this non symmetric operad are the edges of T , and the operations are freely generated by the vertices of T , i.e. if $\sigma = (e_1, e_2, \dots, e_n; e)$ is an ordered sequence of edges of T and there is a vertex v with incoming edges e_1, \dots, e_n in this order and outgoing edge e , then $\Omega^\pi(T)(\sigma) = \{v\}$. One can then “compose” vertices, indicated by the tree T . Hence a map $R \rightarrow T$ in Ω^π is simply a planar operad map $\Omega^\pi(R) \rightarrow \Omega^\pi(T)$. We observe that if $f: R \rightarrow T$ is an isomorphism, then the planar operad structures imply that R and T have the same planar shape and they differ only on the names of their edges and vertices. To avoid dealing with these irrelevant isomorphisms, further on we will replace Ω^π by a skeleton of it, and call this new category Ω^π as well. With this new convention, we observe that all the maps of Ω^π are generated by two types, *faces* and *degeneracies*. These types of maps generalise the corresponding notions in the category Δ defining simplicial sets, in the following way. Let L_n denote the linear tree with n vertices, $n \geq 0$:



If we consider the categorical definition of Δ , we observe that the category

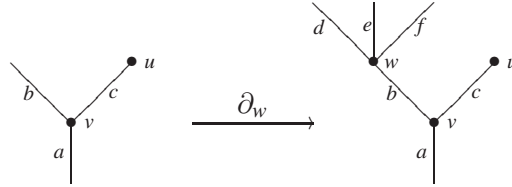
$$[n] = 0 \leftarrow 1 \leftarrow 2 \leftarrow \dots \leftarrow n$$

is in fact $[n] = \Omega(L_n)$, hence Δ is fully faithfully embedded into Ω^π by $[n] \mapsto L_n$.

The face maps in Ω^π are all those monic operad maps $\partial: \Omega^\pi(R) \rightarrow \Omega^\pi(T)$ which increase the number of vertices by one (i.e. $|\text{Vert}(T)| = |\text{Vert}(R)| + 1$) and the degeneracies are all those epic operad maps $\sigma: \Omega^\pi(T) \rightarrow \Omega^\pi(R)$ which decrease the number of vertices by one.

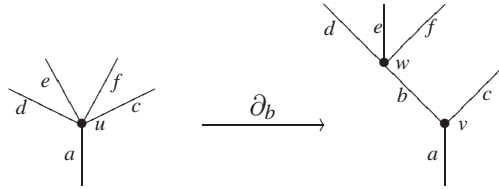
It follows that face maps can be of the following types:

(a) the following picture is an example of an *outer face*



which is just an inclusion of operads;

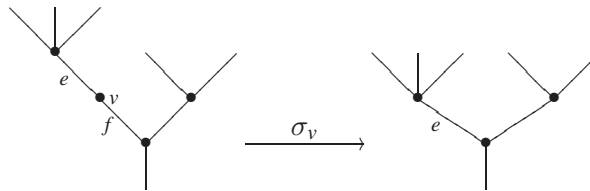
(b) the following picture is an example of an *inner face*



where $\partial_b: \Omega^\pi(R) \longrightarrow \Omega^\pi(T)$ is the identity on the objects (edges), and sends the operation $u \in \Omega^\pi(R)(d, e, f, c; a)$ to the composite operation $v \circ_1 w \in \Omega^\pi(T)(d, e, f, c; a)$, which we can denote without ambiguity by $v \circ_b w$.

Note that the seemingly special cases of face maps into the corolla Cor_n , $n \geq 2$ fall under case (a): these face maps are all the $n + 1$ possible edge inclusions of the trivial tree $|$ to Cor_n .

On the other hand, a degeneracy always looks like



where both of the objects e, f are sent to e , the operation v to the identity operation id_e and σ_v is the identity elsewhere.

We will use the following terminology with respect to faces and degeneracies:

- If e is an inner edge of a tree T , then T/e denotes the tree resulting from T by contracting e . The inner face corresponding to this contraction is usually denoted by $\partial_e: T/e \longrightarrow T$.
- If v is an outer vertex of a tree T (that is, it has exactly one inner edge adjacent to it), then T/v denotes the tree resulting from T by removing v and all the external edges adjacent to it (with the obvious choice for the root of T/v when

one of these external edges happens to be the root of T). We call this procedure “cutting vertex v ”. The outer face correspondig to cutting v is usually denoted by $\partial_v: T/v \longrightarrow T$.

- If v is a vertex of valence one of a tree T then $T \setminus v$ denotes the tree resulting from T by removing v . The degeneracy corresponding to this removal is usually denoted by $\sigma_v: T \longrightarrow T \setminus v$.

The category Ω is obtained from Ω^π via the functor Symm . The objects of Ω are (non planar) trees and the arrows $R \longrightarrow T$ are operad maps $\text{Symm}(\Omega(\bar{R})) \longrightarrow \text{Symm}(\Omega(\bar{T}))$, where \bar{T} denotes a planar representative of T . One can check that the resulting operad does not depend on the chosen representatives, hence the definition makes sense. Later on we will use this independence from chosen representatives: we often describe the operad $\Omega(T)$ by picking a representative \bar{T} and giving only the description of the planar operad $\Omega^\pi(\bar{T})$.

The definition given above implies that there is an extra type of generator for the maps in Ω , namely the isomorphisms.

The category of dendroidal sets is the presheaf category on Ω :

$$dSets := Sets^{\Omega^{\text{op}}} = \text{Funct}(\Omega^{\text{op}}, Sets).$$

If X is a dendroidal set, the elements of X_T are called *dendrices of shape T* . The *representable dendroidal set* associated to a tree T is the functor

$$\Omega[T] := \Omega(-, T): \Omega^{\text{op}} \longrightarrow Sets.$$

By the Yoneda lemma, a dendrex $t \in X_T$ is the same thing as a map of dendroidal sets $\Omega[T] \longrightarrow X$. The Yoneda lemma in general is a very useful tool in the theory of simplicial- and dendroidal sets, allowing us to swap between maps and dendrices whenever needed. We are going to exploit this property in the following without mentioning it. A first application of the Yoneda lemma in the dendroidal setting proves that every dendroidal set is a colimit of representable ones, a property generalising the well known fact for simplicial sets.

For any given tree T one can define some dendroidal subsets of the representable $\Omega[T]$, like the boundary $\partial\Omega[T]$ or the inner horn $\Lambda^e[T]$ with respect to the inner edge e . Dendroidal sets are analogous to simplicial sets in many ways. For example, inner horns can be used to define *inner Kan complexes* in the category of dendroidal sets: we say that a dendroidal set X *satisfies the inner Kan condition* if for any inner horn $h: \Lambda^e[T] \longrightarrow X$ there exists a dendrex $t: \Omega[T] \longrightarrow X$ such that the following diagram

commutes:

$$\begin{array}{ccc} \Lambda^e[T] & \xrightarrow{h} & X \\ \downarrow & \nearrow i & \\ \Omega[T] & & \end{array}$$

In this case X is called an *inner Kan complex* or a *quasi-operad*, analogously to the simplicial case where an inner Kan complex was called by A. Joyal a *quasi-category*.

Another notion that generalises from simplicial sets and categories to dendroidal sets and operads is the nerve functor. The *dendroidal nerve* $N_d: \mathcal{Op} \rightarrow dSets$ can be defined by setting for any operad P

$$(N_d(P))_T := \mathcal{Op}(\Omega(T), P).$$

In the next few lines we introduce the notions of *k-skeleton* and *k-coskeleton* of a dendroidal set. For every $k \in \mathbb{N}$ let Ω_k denote the full subcategory of Ω consisting of trees with at most k vertices. The presheaf category on Ω_k is called the category of *k-truncated dendroidal sets* and is denoted by $dSets_k$. The inclusion $i_k: \Omega_k \rightarrow \Omega$ induces the truncation functor $i_k^*: dSets \rightarrow dSets_k$ which has a left adjoint $(i_k)_!$ and a right adjoint $(i_k)_*$. It follows that the composites

$$(i_k)_! i_k^*, (i_k)_* i_k^*: dSets \rightarrow dSets$$

form an adjoint pair of endofunctors. The left adjoint $(i_k)_! i_k^*$ is usually denoted by Sk_k and is called the *k-skeleton* functor. The right adjoint is denoted by coSk_k and is called the *k-coskeleton* functor.

A dendroidal set X is said to be *k-coskeletal* if the canonical map $X \rightarrow \text{coSk}_k(X)$ is an isomorphism. Another way to state this is that for every dendroidal set Y and every map of dendroidal sets $\phi: \text{Sk}_k(Y) \rightarrow X$ there exists a unique extension

$$\begin{array}{ccc} \text{Sk}_k(Y) & \xrightarrow{\phi} & X \\ \downarrow & \nearrow \exists! & \\ Y & & \end{array}$$

Since any $Y \in dSets$ is a colimit of representables, we can infer that if the previous statement holds for all $Y = \Omega[T]$ where T is any tree with $k + 1$ vertices, then it holds in general. In this case $\text{Sk}_k(Y) = \text{Sk}_k(\Omega[T]) = \partial\Omega[T]$. Note that in view of the Yoneda lemma we can think of the composite

$$\text{Sk}_k(\Omega[T]) \rightarrow \Omega[T] \xrightarrow{t} X$$

as the boundary- or k -skeleton of the dendrex t . To emphasize this point of view, sometimes we will denote this composite by $\text{Sk}_k(t)$.

One can define the dual notion of k -skeletal dendroidal sets similarly.

Remark 2.1 Note that the dendroidal definition of the functors Sk_k and coSk_k uses a filtration of the objects of Ω by the number of the vertices of trees. Later on, we will use the term *degree* to refer to this natural number.

2.2 A closed symmetric monoidal category structure on dendroidal sets

Since $dSets$ is a presheaf category, it can be endowed with the usual cartesian closed category structure present in any presheaf category. There is another interesting symmetric monoidal structure on $dSets$ that will prove to be useful in the definition of dendroidal weak n -categories of Section 3. Our goal is to recall this monoidal structure in the current section, together with those properties that will be used. For more details on this subject one can consult [11, 12, 14].

One way to define the mentioned monoidal structure on $dSets$ is by transferring the Boardman-Vogt monoidal structure on \mathcal{Op} , via the dendroidal nerve functor. We adopt this road, and we start by recalling the Boardman-Vogt tensor product for symmetric operads (a generalisation of the B-V tensor product for classical operads in [4]).

Let $P, Q \in \mathcal{Op}$. We define a new operad, $P \otimes Q$, as follows. The set of objects is $ob(P \otimes Q) := ob(P) \times ob(Q)$ and we denote the elements of this set by $a \otimes x := (a, x)$. We describe the operations of $P \otimes Q$ in terms of generators and relations. There are two types of generators:

- (a) For any $p \in P(a_1, \dots, a_n; a)$ and any $x \in ob(Q)$,

$$p \otimes x \in P \otimes Q(a_1 \otimes x, \dots, a_n \otimes x; a \otimes x).$$

- (b) For any $a \in ob(P)$ and any $q \in Q(x_1, \dots, x_m; x)$,

$$a \otimes q \in P \otimes Q(a \otimes x_1, \dots, a \otimes x_m; a \otimes x).$$

The relations also are of two types:

- (a) Relations that imply precisely that the obvious maps

$$P \xrightarrow{\text{id} \otimes x} P \otimes Q \text{ and } Q \xrightarrow{a \otimes \text{id}} P \otimes Q$$

are maps of operads for any fixed $x \in ob(P), a \in ob(Q)$.

- (b) For any $p \in P(a_1, \dots, a_n; a)$ and $q \in Q(x_1, \dots, x_m; x)$ the following two operations are the same in $P \otimes Q$

where $\sigma_{n,m} \in \Sigma_{n+m}$ denotes the permutation that makes the order of the inputs on the right-hand side of the equation the same as the order of the inputs on the left-hand side.

The tensor product we defined is a bifunctor $- \otimes - : \mathcal{Op} \times \mathcal{Op} \longrightarrow \mathcal{Op}$ and it induces a symmetric closed monoidal category structure on \mathcal{Op} . The right adjoint of any functor $- \otimes Q$ is denoted by $\underline{\mathcal{Op}}(Q, -) : \mathcal{Op} \longrightarrow \mathcal{Op}$. In particular, $\underline{\mathcal{Op}}(Q, \mathcal{Sets})$ is the operad of Q -algebras. (For the definition of $\underline{\mathcal{Op}}(Q, -)$ see [14].)

We can now make use of the functor $N_d : \mathcal{Op} \longrightarrow dSets$ to transfer the Boardman-Vogt tensor product to dendroidal sets:

- For any two representable dendroidal sets $\Omega[T]$ and $\Omega[R]$, define

$$\Omega[T] \otimes \Omega[R] := N_d(\Omega(T) \otimes \Omega(R)).$$

- Extend the definition cocontinuously, i.e. for any $X, Y \in dSets$ write $X = \text{colim}_T \Omega[T]$, $Y = \text{colim}_R \Omega[R]$ as colimits of representables and define

$$X \otimes Y := \text{colim}_{T,R} \Omega[T] \otimes \Omega[R].$$

The bifunctor $- \otimes - : dSets \times dSets \longrightarrow dSets$ induces a symmetric closed monoidal structure on $dSets$, the right adjoint of $- \otimes Y$ is the functor

$$\underline{dSets}(Y, -) : dSets \longrightarrow dSets,$$

given on objects (by Yoneda lemma) by

$$\underline{dSets}(Y, Z)_T = dSets(Y \otimes \Omega[T], Z).$$

The following properties will prove to be useful in Section 4:

Proposition 2.2 (Lemma 4.3.3 in [14]) For any operad $P \in \mathcal{Op}$ and for any tree $T \in \Omega$

$$N_d(P) \otimes \Omega[T] \simeq N_d(P \otimes \Omega(T)).$$

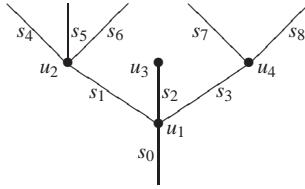
Proposition 2.3 (Corollary 9.3 in [12]) For all operads $P, Q \in \mathcal{Op}$

$$\underline{dSets}(N_d(P), N_d(Q)) \simeq N_d(\underline{\mathcal{Op}}(P, Q)).$$

2.3 The dendroidal Grothendieck construction

The aim of this Section is to provide an ingredient we are going to use in the description of dendroidal weak higher categories. The data we start with is a functor $X: \mathbb{S}^{\text{op}} \longrightarrow dSets$ where \mathbb{S} is a cartesian category, and we are going to assign to X a new dendroidal set $\int_{\mathbb{S}} X$, called *the Grothendieck construction of X* .

To achieve this goal, we need some preliminary definitions. Since \mathbb{S} is cartesian it is an operad, hence it makes sense to talk about the dendroidal set $N_d(\mathbb{S})$. Suppose that for a fixed tree T , $t \in N_d(\mathbb{S})$ is a dendrex of shape T . That is, t intuitively looks like the tree T decorated with objects and operations of the operad \mathbb{S} :



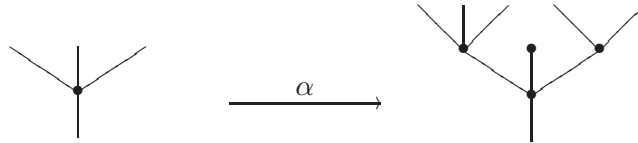
where the s_i are objects of \mathbb{S} , and – for example – $u_1: s_1 \times s_2 \times s_3 \longrightarrow s_0$ is a map in \mathbb{S} . To such a t we can assign an object of \mathbb{S} , called $\text{in}(t)$, which is the cartesian product of the objects labeling the leaves of T : since $t \in \mathcal{O}p(\Omega(T), \mathbb{S})$,

$$\text{in}(t) := \prod_{l \in \text{Leaves}(T)} t(l).$$

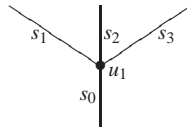
Furthermore, we can assign to a $t \in N_d(\mathbb{S})_T$ and a map $\alpha: R \longrightarrow T$ of Ω a map in \mathbb{S}

$$\text{in}(\alpha): \text{in}(t) \longrightarrow \text{in}(\alpha^* t)$$

by first composing the maps of \mathbb{S} , indicated by t and α , and then taking the product. For example, if $\alpha: R \longrightarrow T$ is the inclusion to the root vertex (in this case a composite of three outer faces)



and t is as above, then $\alpha^* t$ is



and $\text{in}(\alpha) = u_2 \times u_3 \times u_4$. In particular, if α is an inner face or a degeneracy then $\text{in}(\alpha)$ is the identity map of $\text{in}(t) = \text{in}(\alpha^*t)$, and if $R' \xrightarrow{\beta} R \xrightarrow{\alpha} T$ are maps of Ω then $\text{in}(\alpha\beta) = \text{in}(\beta)\text{in}(\alpha)$.

In view of the definitions above we can define $\int_{\mathbb{S}} X$ as follows. The set $(\int_{\mathbb{S}} X)_T$ consists of pairs (t, x) where $t \in N_d(\mathbb{S})_T$ and

$$x: \Omega[T] \longrightarrow \prod_{s \in \mathbb{S}} X(s)$$

is a degree preserving map such that $x(r) \in X(\text{in}(r^*t))$ for any $r \in \Omega[T]_R$. There is one more condition on x : it has to be compatible with the dendroidal structure of the various dendroidal sets involved. Explicitly, for a chain of arrows $R' \xrightarrow{\alpha} R \xrightarrow{r} T$ in Ω we have $r \in \Omega[T]_R$ and $\alpha^*r = r\alpha \in \Omega[T]_{R'}$, hence

$$x(r) \in X(\text{in}(r^*t))_R \quad \text{and} \quad x(\alpha^*r) \in X(\text{in}((r\alpha)^*t))_{R'}.$$

The data above also induces two maps

$$\begin{array}{ccc} X(\text{in}(r^*t))_R & & X(\text{in}((r\alpha)^*t))_{R'} \\ & \searrow \alpha^* & \swarrow X(\text{in } \alpha) \\ & X(\text{in}(r^*t))_{R'} & \end{array}$$

We require

$$(2-1) \quad \alpha^*(x(r)) = X(\text{in } \alpha)(x(\alpha^*r)).$$

The dendroidal structure on $\int_{\mathbb{S}} X$ is defined as follows. Suppose that $\delta: R \longrightarrow T$ is a map in Ω and $(t, x) \in (\int_{\mathbb{S}} X)_T$ a dendrex of shape T . The map δ induces the map of dendroidal sets $\Omega[\delta]: \Omega[R] \longrightarrow \Omega[T]$. We define

$$(2-2) \quad \delta^*(t, x) := (\delta^*t, x \circ \Omega[\delta]).$$

One can check that with this structure $\int_{\mathbb{S}} X$ is indeed a dendroidal set. The following theorem and proposition collect two important properties of the dendroidal Grothendieck construction.

Theorem 2.4 ([11, 14]) *Let $X: \mathbb{S}^{\text{op}} \longrightarrow dSets$ be a diagram of dendroidal sets. If for all $s \in \mathbb{S}$ every $X(s)$ is an inner Kan complex then so is $\int_{\mathbb{S}} X$.*

Proposition 2.5 *Let $X: \mathbb{S}^{\text{op}} \longrightarrow dSets$ be a diagram of dendroidal sets and $k \geq 2$ a natural number. If $X(s)$ is k -coskeletal for every $s \in \mathbb{S}$ then so is $\int_{\mathbb{S}} X$.*

Proof Let us start with the remark that $k \geq 2$ is needed because dendroidal nerves of operads are 2-coskeletal (a generalisation of the well known fact for nerves of categories, proven in [11, 14]).

Our task is to prove that, for any tree T with $k + 1$ vertices, every map of dendroidal sets $\phi: \partial\Omega[T] \rightarrow \int_{\mathbb{S}} X$ extends uniquely as

$$\begin{array}{ccc} \partial\Omega[T] & \xrightarrow{\phi} & \int_{\mathbb{S}} X \\ \downarrow & \nearrow \exists! & \\ \Omega[T] & & \end{array}$$

We suppose existence and prove uniqueness first. Let $(t_1, x_1), (t_2, x_2) \in (\int_{\mathbb{S}} X)_T$ be two dendrices filling the boundary ϕ . The dendroidal set $N_d(\mathbb{S})$ is k -coskeletal since $k \geq 2$. Hence by equation (2–2) we can infer that $t_1 = t_2 = t$. Let $u: R \rightarrow T$ be a face. Since $u^*(t, x_1) = u^*(t, x_2)$, we obtain $x_1 \circ \Omega[u] = x_2 \circ \Omega[u]$. On the other hand,

$$x_i(u) = (x_i \circ \Omega[u])(\text{id}_R)$$

for $i = 1, 2$, implying $x_1(u) = x_2(u)$. We can use now equation (2–1) for $r = \text{id}_T$ and $\alpha = u$ to conclude that $u^*(x_1(\text{id}_T)) = u^*(x_2(\text{id}_T))$ as dendrices of shape R in $X(\text{in}(t))$. Since this is true for any face $u: R \rightarrow T$ and $X(\text{in}(t))$ is k -coskeletal, it follows that also $x_1(\text{id}_T) = x_2(\text{id}_T)$. We can infer that $x_1 = x_2$, thus the filler is unique.

The argument above also contains the information how to construct a filler (t, x) of ϕ , giving a proof of the existence of such an extension. \square

Remark 2.6 If we restrict our attention to dendroidal sets where the only nontrivial dendrices are of linear shapes, Proposition 2.5 implies that the same property is true for simplicial sets and the simplicial Grothendieck construction.

2.4 The homotopy coherent dendroidal nerve of an operad

Let \mathcal{E} be a symmetric monoidal model category with an *interval* H : that is, an object H of \mathcal{E} , together with two *points* $0: I \rightarrow H$ and $1: I \rightarrow H$, an augmentation $\epsilon: H \rightarrow I$ and an associative binary operation $\vee: H \otimes H \rightarrow H$ for which 0 is unital and 1 is absorbing, satisfying $\epsilon 1 = \epsilon 0 = \text{id}$. In this case one can modify the nerve construction for operads enriched in \mathcal{E} in such a way that the resulting dendroidal set encodes also the homotopies in the operad.

An interesting example of such a situation is when \mathcal{E} is the category of categories with the usual cartesian product, the folk model structure and the interval H is the category

$$0 \xrightarrow{\simeq} 1$$

with two objects and one isomorphism between them. The required structure on H is the obvious one: 0 is the neutral element, 1 is the absorbing one, and the rest of the interval structure on H is completely determined by the previous choices. Indeed, since the unit of the monoidal structure is the terminal object in \mathcal{E} (the category $*$ with one object and no other morphisms than the identity), the counit $\epsilon: H \rightarrow *$ is obvious. The various compatibility conditions imply that the monoid structure $\vee: H \times H \rightarrow H$ is given by “the maximum operation”: on the objects, $i \vee j = \max\{i, j\}$.

Since we are interested only in this example, from now on \mathcal{E} denotes the category of categories with the structure mentioned above, although everything can be carried out similarly in the general case.

Let us denote the category of operads enriched in \mathcal{E} by $\mathcal{Op}_{\mathcal{E}}$. The functor

$$\mathrm{hcN}_d: \mathcal{Op}_{\mathcal{E}} \rightarrow dSets$$

is defined by

$$\mathrm{hcN}_d(P)_T := \mathcal{Op}_{\mathcal{E}}(W(\Omega(T)), P)$$

where W is the W -construction for coloured operads (see [3] for details) and $\Omega(T)$ is the discrete version in \mathcal{E} of the operad induced by the tree T . We will need later an explicit description of $W(\Omega(T))$, hence we give it here.

Recall that for a tree T

$$\Omega(T) = \mathrm{Symm}(\Omega^{\pi}(\tilde{T}))$$

where $\mathrm{Symm}: \mathcal{Op}_{\mathcal{E}}^{\pi} \rightarrow \mathcal{Op}_{\mathcal{E}}$ is the \mathcal{E} -enriched version of the symmetrization functor from non symmetric operads to operads, and \tilde{T} is any planar representative of T . Moreover, the W -construction commutes with Symm , thus

$$W\Omega(T) = \mathrm{Symm}(W\Omega^{\pi}(\tilde{T})).$$

This property allows us to describe $W\Omega(T)$ by using an arbitrary planar representative of T . The objects of $W\Omega^{\pi}(\tilde{T})$ are the edges of T . Suppose that $\sigma = (e_1, e_2, \dots, e_n; e)$ is an ordered sequence of objects. We can distinguish two cases for the category of operations corresponding to σ :

- (1) If $\Omega^{\pi}(\tilde{T})(\sigma) = \emptyset$ – the empty category – then also $W\Omega^{\pi}(\tilde{T})(\sigma) = \emptyset$.

- (2) If $\Omega^\pi(\bar{T})(\sigma) \neq \emptyset$, it follows that \bar{T} has a subtree \bar{T}_σ with leaves e_1, \dots, e_n and root e . The set of internal edges of \bar{T}_σ is denoted by $\text{InEdg}(\bar{T}_\sigma)$. From the W -construction it follows then that

$$W\Omega^\pi(\bar{T})(\sigma) = \prod_{f \in \text{InEdg}(\bar{T}_\sigma)} H,$$

where in case the product is empty the result is the unit object of the monoidal structure, which is the category $*$ with one object and no other morphism than the identity.

We still need to define the composition maps in the operad $W\Omega^\pi(\bar{T})$. Suppose that $\sigma = (e_1, \dots, e_n; e)$ and $\rho = (f_1, \dots, f_m; e_i)$ are ordered sequences of edges of T , such that neither $\Omega^\pi(\bar{T})(\sigma)$, nor $\Omega^\pi(\bar{T})(\rho)$ is the empty category. It follows that \bar{T} has subtrees \bar{T}_σ and \bar{T}_ρ , the sets of internal edges of these trees are disjoint and the tree $\bar{T}_{\sigma \circ_i \rho}$ obtained by grafting along the edge e_i has one more internal edge than the previous two together. Let us denote these sets of internal edges by $\text{int}(\sigma)$, $\text{int}(\rho)$ and $\text{int}(\sigma \circ_i \rho)$ respectively. The composition map

$$\circ_i: W\Omega^\pi(\bar{T})(\sigma) \times W\Omega^\pi(\bar{T})(\rho) \longrightarrow W\Omega^\pi(\bar{T})(\sigma \circ_i \rho)$$

is given by

$$\left(\prod_{\text{int}(\sigma)} H \right) \times \left(\prod_{\text{int}(\rho)} H \right) \simeq \left(\prod_{\text{int}(\sigma) \cup \text{int}(\rho)} H \right) \times * \xrightarrow{\text{id} \times 1} \prod_{\text{int}(\sigma \circ_i \rho)} H,$$

where the functor $1: * \longrightarrow H$ is the absorbing element of H .

This concludes the description of the operad $W\Omega(T)$. Note that we still need to mention how the dendroidal structure on $\text{hcN}_d(P)$ is defined. If $\delta: R \longrightarrow T$ is a face map in Ω then it induces a map of operads $\delta: W\Omega(R) \longrightarrow W\Omega(T)$ via the neutral element functor $0: * \longrightarrow H$. In case δ is a degeneracy, the induced functor is obtained by the monoid structure $\vee: H \times H \longrightarrow H$. These definitions are functorial, hence they induce a dendroidal structure on $\text{hcN}_d(P)$.

3 Dendroidal weak n -categories

For any set A there exists a planar operad As_A^π whose algebras are small categories with set of objects A . The objects of As_A^π are ordered pairs $(a_1, a_2) \in A \times A$, and the sets of operations are defined by

$$\begin{aligned} As_A^\pi(\quad; (a, a)) &= *, \\ As_A^\pi((a_1, a_2), (a_2, a_3), \dots, (a_{n-1}, a_n); (a_1, a_n)) &= * \end{aligned}$$

and in all the other cases the set of operations is empty (those ordered sequences $\sigma = (c_1, c_2, \dots, c_n; c)$ of objects of $A \times A$ for which $As_A^\pi(\sigma)$ is not empty will be called *admissible*).

Let $\alpha: As_A^\pi \rightarrow Sets$ be a map of operads. The data-part of such an α determines for any $(a_1, a_2) \in A \times A$ a set $\mathcal{A}(a_1, a_2)$ and for any admissible signature $\sigma = ((a_1, a_2), (a_2, a_3), \dots, (a_{n-1}, a_n); (a_1, a_n))$ a function

$$comp_\sigma: \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) \times \dots \times \mathcal{A}(a_{n-1}, a_n) \longrightarrow \mathcal{A}(a_1, a_n)$$

which in the particular case of $\sigma = (; (a, a))$ is a function $*$ $\longrightarrow \mathcal{A}(a, a)$. The compatibility-part of such an α ensures that the various functions $comp_\sigma$ fit nicely to define units and compositions of arrows in a category \mathcal{A} with object set A . Indeed, we arrive to the conclusion that the relevant ordered sequences are of the type $((a_1, a_2), (a_2, a_3); (a_1, a_3))$ and $(; (a, a))$, etc.

Since the forgetful functor $U: \mathcal{Op} \rightarrow \mathcal{Op}^\pi$ is right adjoint to the symmetrization functor, we infer that the algebras of the operad $As_A := \text{Symm}(As_A^\pi)$ are categories with set of objects A as well.

Remark 3.1 Note that in the description of As_A^π -algebras given above we used the unconventional “left-to-right” composition order for arrows, i.e.

$$\mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) \longrightarrow \mathcal{A}(a_1, a_3)$$

instead of the conventional

$$\mathcal{A}(a_2, a_3) \times \mathcal{A}(a_1, a_2) \longrightarrow \mathcal{A}(a_1, a_3)$$

To avoid unnecessary complications in the future, arising only from notation, whenever we need to give such a composition map associated to some signature σ , we will always stick to the order determined by σ , thus the unconventional order. However, when it is required to give extra details with explicit composites of maps, we will use the conventional “right-to-left” order.

Let X be a dendroidal set and define the functor

$$Cat(X)_- : Sets^{\text{op}} \longrightarrow dSets, \quad Cat(X)_A := \underline{dSets}(N_d(As_A), X).$$

The dendroidal set of *categories enriched in X* (see [11]) is by definition the Grothendieck construction of $Cat(X)_-$. We denote it by

$$Cat(X) := \int_{Sets} Cat(X)_-.$$

One can iterate the process above to obtain a definition of the dendroidal set of n -categories enriched in X :

$$\begin{aligned}\mathcal{Cat}^0(X) &:= X, \\ \mathcal{Cat}^n(X) &:= \mathcal{Cat}(\mathcal{Cat}^{n-1})(X).\end{aligned}$$

To see why this definition is plausible, one can try particular choices of X . For example if $X = N_d(\mathcal{Sets})$, we can prove inductively that $\mathcal{Cat}^n(X)$ is the dendroidal nerve of strict n -categories with the classical definition (see also Example 4.5.5 in [14]). Indeed, for $n = 1$

$$\begin{aligned}\mathcal{Cat}(N_d(\mathcal{Sets})) &= \int_{A \in \mathcal{Sets}} \underline{dSets}(N_d(As_A), N_d(\mathcal{Sets})) \\ &\simeq \int_{A \in \mathcal{Sets}} N_d(\underline{\mathcal{Op}}(As_A, \mathcal{Sets})) \\ &\simeq \int_{A \in \mathcal{Sets}} N_d(\mathcal{Categ}_A) \\ &\simeq N_d(\mathcal{Categ}),\end{aligned}$$

where \mathcal{Categ} denotes the usual monoidal category of small categories, viewed as an operad. The second part of the inductive proof is similar (one uses that for any monoidal category \mathcal{M} , $\underline{\mathcal{Op}}(As_A, \mathcal{M}) \simeq \mathcal{Categ}_A(\mathcal{M})$, where the right-hand side denotes the monoidal category of categories enriched in \mathcal{M} , with set of object A).

We are interested here in another choice for X , which yields the dendroidal definition of weak n -categories: it is plausible to define $X := \text{hc}N_d(\mathcal{Ctg})$ where \mathcal{Ctg} is the category of small categories enriched in \mathcal{E} . (Recall from Section 2.4 that \mathcal{E} denotes the symmetric monoidal model category of categories, together with the interval H . Hence the set of functors between two fixed categories is a category with natural transformations as maps.)

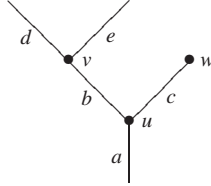
Definition 3.2 ([11]) The dendroidal set of *weak n -categories* is defined as follows:

$$\begin{aligned}w\mathcal{Cat}^0 &:= N_d(\mathcal{Sets}), \\ w\mathcal{Cat}^n &:= \mathcal{Cat}^{n-1}(\text{hc}N_d(\mathcal{Ctg})) \quad \text{for } n > 0.\end{aligned}$$

The rest of this Section is dedicated to the study of coskeletality of weak n -categories. The first result is

Lemma 3.3 If $T \in \Omega$ is a tree with 3 vertices and $t, s \in w\mathcal{Cat}_T^1$ satisfy $\text{Sk}_2(t) = \text{Sk}_2(s)$ then $t = s$.

Proof To illustrate our argument better, we will work with a chosen tree, the general case can be carried out in the same way. So let $T \in \Omega$ be the tree with a planar representative as below.



Let $x \in w\mathcal{Cat}_T^1$ be a dendrex of shape T , that is a map of operads enriched in \mathcal{E} , $x : W\Omega(T) \rightarrow \mathcal{C}tg$. Let us adopt the notations of Section 2.4. It follows that x consists of compatible functors $x_\sigma : H^{\text{int}(\sigma)} \rightarrow \mathcal{C}tg(\sigma)$ and the only functor we have to describe in terms of the 2-skeleton of x is the one corresponding to $\sigma = (d, e; a)$ (the other functors lie in the image of $\text{Sk}_2(x)$). In this case the domain of x_σ is the groupoid $H^2 = H_c \times H_b$, represented as the square

$$\begin{array}{ccc} (0, 0) & \xrightarrow{(\text{id}_0, b)} & (0, 1) \\ (c, \text{id}_0) \downarrow & & \downarrow (c, \text{id}_1) \\ (1, 0) & \xrightarrow{(\text{id}_1, b)} & (1, 1) \end{array}$$

where we think of the copy of H corresponding to an internal edge f as the groupoid $H_f = 0 \xrightarrow{f} 1$. Since the domain is a groupoid, we observe that if x_σ is already defined on a “connected part” of the “square” $H_c \times H_b$, then it is defined on the “convex hull” of that component. We conclude that in order to know x_σ , it is enough to know its image on the sets of arrows $\text{Opd} = \{(c, \text{id}_1), (\text{id}_1, b)\}$ and $\text{Face} = \{(c, \text{id}_0), (\text{id}_0, b)\}$.

To conclude the proof, first we show that since x is a map of operads, $x_\sigma(\text{Opd})$ is determined by $\text{Sk}_2(x)$. Indeed, the commutative square

$$\begin{array}{ccc} H_c \times \{v\} & \xrightarrow{x \times x} & \mathcal{C}tg(b; a) \times \mathcal{C}tg(d, e; b) \\ \circ_b \downarrow & & \downarrow \circ_b \\ H_c \times H_b & \xrightarrow{x_\sigma} & \mathcal{C}tg(d, e; a) \end{array}$$

implies that we know x_σ on the arrow (c, id_1) , and a similar square gives the image of (id_1, b) .

Second, we show that $x_\sigma(\text{Face})$ is in the image of $\text{Sk}_2(x)$. We observe that the inner faces $\partial_b : R \rightarrow T$ and $\partial_c : R' \rightarrow T$, according to the definition of the dendroidal structure on $w\mathcal{Cat}^1$, induce enriched operad maps $W\Omega(R) \rightarrow W\Omega(T)$

and $W\Omega(R') \longrightarrow W\Omega(T)$ respectively. Each of these maps has in its image the corresponding element of $Dend$, hence $x(Face)$ is in the image of $Sk_2(x)$. \square

Proposition 3.4 *Let $T \in \Omega$ be a tree such that $|\text{Vert}(T)| \geq 3$. If $t, s \in wCat_T^1$ satisfy $Sk_2(t) = Sk_2(s)$ then $t = s$.*

Proof We proceed by induction on $n = |\text{Vert}(T)|$, the case $n = 3$ is covered in Lemma 3.3. Suppose that $x : W\Omega(T) \longrightarrow Ctg$ is a dendrex of shape T . First we notice that we only need to describe the functor $x_\sigma : H^{\text{int}(\sigma)} \longrightarrow Ctg(\sigma)$ in terms of the $Sk_{n-1}(x)$ where σ is the ordered sequence of colours $(\text{Leaves}(\bar{T}); \text{root}(\bar{T}))$ for a chosen planar representative \bar{T} . (The other components of x are already contained in the image of $Sk_{n-1}(x)$.)

The domain of x_σ is a groupoid with the shape of an n -cube, having in its vertices the trivial categories $(\epsilon_1, \dots, \epsilon_n)$, $\epsilon_i \in \{0, 1\}$. Denote by \bar{H}_k the full subcategory of $H^{\text{int}(\sigma)}$ spanned by the categories $(\epsilon_1, \dots, \epsilon_{k-1}, 1, \epsilon_{k+1}, \dots, \epsilon_n)$, $\epsilon_i \in \{0, 1\}$ (one of the hyperfaces of the n -cube, containing the vertex $(1, 1, \dots, 1)$). Denote by ϕ_k the arrow $(0, 0, \dots, 0) \longrightarrow (0, \dots, 0, 1, 0, \dots, 0)$ of $H^{\text{int}(\sigma)}$ (one of the edges of the n -cube, starting in $(0, 0, \dots, 0)$). Define the sets

$$Opd := \{\bar{H}_k | k = 1, 2, \dots, n\} \quad \text{and} \quad Face := \{\phi_k | k = 1, 2, \dots, n\}.$$

Since the “convex hull” of $Opd \cup Face$ is the whole domain of x_σ , it is enough to prove that $x_\sigma(Opd)$ is completely determined by $Sk_{n-1}(x)$ and $x_\sigma(Face)$ is in the image of $Sk_k(x)$. Both of these assertions are true, by similar arguments to the ones in the proof of Lemma 3.3. \square

The following propositions of [14] and [11] helps us in proving that $wCat^1$ is 3-coskeletal.

Proposition 3.5 *(Proposition 3.2.5 in [14]) Let X be a dendroidal set and $k \geq 2$ an integer. If X satisfies the strict inner Kan condition for all trees T of degree at least k , then X is k -coskeletal.*

Proposition 3.6 *(Proposition 7.2 in [11]) Let P be a locally fibrant operad in \mathcal{E} (that is, for any ordered sequence of objects $\sigma = (c_1, \dots, c_n; c)$ the category $P(\sigma)$ is fibrant with respect to the folk model structure). Then $hcN_d(P)$ is an inner Kan complex.*

Corollary 3.7 *(Lemma 4.6.3 in [14]) The dendroidal set $wCat^1$ is 3-coskeletal.*

Proof In view of Proposition 3.5 it is enough to prove that $wCat^1$ satisfies the strict inner Kan condition for all trees with $|\text{Vert}(T)| \geq 3$. Let T be such a tree. Theorem 3.6 implies that $wCat^1$ is an inner Kan complex, hence every inner horn $\Lambda^e[T] \longrightarrow wCat^1$ has at least one filler t . Suppose that s is an other filler of the same horn. Since $|\text{Vert}(T)| \geq 3$, it follows that $\text{Sk}_2(t) = \text{Sk}_2(s)$. We conclude thus by Proposition 3.4 that $t = s$. \square

Corollary 3.7 implies that $wCat^n$ is 3-coskeletal for every $n \geq 1$. (Note that $wCat^0$ is already 2-coskeletal.) To prove this, the following lemma is needed.

Lemma 3.8 *If X is a k -coskeletal dendroidal set and Z is an arbitrary dendroidal set then $\underline{dSets}(Z, X)$ is k -coskeletal.*

Proof The goal is to see that for any dendroidal set Y there exists a natural bijection

$$dSets(Y, \underline{dSets}(Z, X)) \simeq dSets(\text{Sk}_k Y, \underline{dSets}(Z, X)).$$

Indeed, once one observes that $\text{Sk}_k(Y \otimes Z) \subseteq (\text{Sk}_k Y) \otimes Z$, one can conclude that there are natural one-to-one correspondences between the following Hom sets:

$$\begin{aligned} dSets(Y, \underline{dSets}(Z, X)) &\simeq dSets(Y \otimes Z, X) \\ &\simeq dSets(Y \otimes Z, \text{coSk}_k X) \\ &\simeq dSets(\text{Sk}_k(Y \otimes Z), X) \\ &\simeq dSets((\text{Sk}_k Y) \otimes Z, X) \\ &\simeq dSets(\text{Sk}_k Y, \underline{dSets}(Z, X)). \end{aligned}$$

\square

Theorem 3.9 *For every $n \geq 1$ the dendroidal set $wCat^n$ is 3-coskeletal.*

Proof We proceed by induction on n . It was proven in Corollary 3.7 that $wCat^1$ is 3-coskeletal. Suppose that $wCat^n$ is 3-coskeletal. It follows from Lemma 3.8 that for any set A , the dendroidal set $\text{Cat}(wCat^n)_A = \underline{dSets}(N_d(As_A), wCat^n)$ is 3-coskeletal. Hence Proposition 2.5 implies that

$$wCat^{n+1} = \int_{Sets} \text{Cat}(wCat^n)_-$$

is 3-coskeletal. \square

4 Dendroidal weak 1- and 2-categories

4.1 Weak 1-categories

We start the Section with the description of the dendroidal set $wCat^1 = \text{hcN}_d(\mathcal{C}tg)$. We can use Corollary 3.7 to come to the conclusion that it is enough to describe the sets $(wCat^1)_T = \mathcal{O}p_{\mathcal{E}}(W\Omega(T), \mathcal{C}tg)$ for trees T with at most 3 vertices. Before we start with the description, let us make a useful notational convention: from now on, given n categories X_1, \dots, X_n and integers $1 \leq i \leq j \leq n$, $(X)_i^j$ will denote the category $X_i \times \dots \times X_j$.

- (1) The first choice of T is the tree $|$. In this case $W\Omega(T) = \Omega(|)$ is the operad on one object and only the identity operation, hence an element of $(wCat^1)_|$ is the same as the choice of a category.
- (2) Let $T = \text{Cor}_n$, the n -corolla. In this case still $W\Omega(\text{Cor}_n) = \Omega(\text{Cor}_n)$, hence an element of $(wCat^1)_{\text{Cor}_n}$ is the same as the choice of $n+1$ categories X_1, \dots, X_n and X , together with a functor $F : (X)_1^n \longrightarrow X$. Note that in case $n = 0$, the \mathcal{E} -enriched operad structure on $\mathcal{C}tg$ implies that $(X)_1^n$ has to be considered the unit of the \mathcal{E} -enriched monoidal category $\mathcal{C}tg$. This unit is the category $*$ on one object and no other arrows than the identity. Hence we infer that a dendrex of shape Cor_0 amounts to the choice of a category X , together with an object of it.
- (3) Let $T = \text{Cor}_n \circ_i \text{Cor}_m$. Let us give a detailed description of maps of operads $\alpha : W\Omega(T) \longrightarrow \mathcal{C}tg$ since this is the first time when the interval H plays a role in the definition of the operad $W\Omega(T)$. So far it is clear that, as in cases (1) and (2), such an α determines
 - (3a) a choice of $n+1$ categories X_1, \dots, X_n, X together with a functor $F_1 : (X)_1^n \longrightarrow X$;
 - (3b) a choice of m categories Y_1, \dots, Y_m and a functor $F_2 : (Y)_1^m \longrightarrow X_i$.

There is one more building part of such an α , which is a functor

$$H \longrightarrow \mathcal{C}tg((X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n, X).$$

But such a functor contains exactly the same data as the choice of two functors $G, G' : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \longrightarrow X$ and a natural isomorphism $\phi : G \longrightarrow G'$. The only thing we have not covered yet with the investigation of such a dendrex

is that α is a map of operads, which means that the diagram of categories

$$\begin{array}{ccc} * \times * & \xrightarrow{\alpha \times \alpha} & \mathcal{C}tg((X)_1^n, X) \times \mathcal{C}tg((Y)_1^m, X_i) \\ \downarrow \circ_i & & \downarrow \circ_i \\ H & \xrightarrow{\alpha} & \mathcal{C}tg((X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n, X) \end{array}$$

is commutative. One can spell out that this yields to $G' = F_1 \circ_i F_2$. We can conclude thus that the last bit of information α provides is

- (3c) a choice of a functor $G : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \longrightarrow X$ and a natural isomorphism $\phi : G \longrightarrow F_1 \circ_i F_2$.

For the remaining choices of the tree T we give only the result.

- (4) Let $T = \text{Cor}_n \circ_i (\text{Cor}_m \circ_j \text{Cor}_k)$. A map of operads $W\Omega(T) \longrightarrow \mathcal{C}tg$ is the same as
- (4a) a choice of $n + 1$ categories X_1, \dots, X_n, X together with a functor $F_1 : (X)_1^n \longrightarrow X$;
 - (4b) a choice of m categories Y_1, \dots, Y_m and a functor $F_2 : (Y)_1^m \longrightarrow X_i$;
 - (4c) a choice of k categories Z_1, \dots, Z_k and a functor $F_3 : (Z)_1^k \longrightarrow Y_j$;
 - (4d) a choice of a functor $G_1 : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \longrightarrow X$ and a natural isomorphism $\phi_1 : G_1 \longrightarrow F_1 \circ_i F_2$;
 - (4e) a choice of a functor $G_2 : (Y)_1^{j-1} \times (Z)_1^k \times (Y)_{j+1}^m \longrightarrow X_i$ and a natural isomorphism $\phi_2 : G_2 \longrightarrow F_2 \circ_j F_3$;
 - (4f) a choice of a functor $K : (X)_1^{i-1} \times (Y)_1^{j-1} \times (Z)_1^k \times (Y)_{j+1}^m \times (X)_{i+1}^n \longrightarrow X$ and two natural isomorphisms

$$\psi_1 : K \longrightarrow F_1 \circ_i G_2, \quad \psi_2 : K \longrightarrow G_1 \circ_{\tilde{j}} F_3$$

where $\tilde{j} = i + j - 1$, such that the following diagram of natural isomorphisms is commutative:

$$\begin{array}{ccc} K & \xrightarrow{\psi_1} & F_1 \circ_i G_2 \\ \psi_2 \downarrow & & \downarrow F_1 \circ_i \phi_2 \\ G_1 \circ_{\tilde{i}} F_3 & \xrightarrow{\phi_1 \circ_{\tilde{j}} F_3} & F_1 \circ_i F_2 \circ_j F_3 \end{array}$$

- (5) Let $T = \text{Cor}_n \circ_{i,j} (\text{Cor}_m, \text{Cor}_k)$ for some $1 \leq i < j \leq n$. A map of operads $W\Omega(T) \longrightarrow \mathcal{C}tg$ is the same as
- (5a) a choice of $n + 1$ categories X_1, \dots, X_n, X together with a functor $F_1 : (X)_1^n \longrightarrow X$;

- (5b) a choice of m categories Y_1, \dots, Y_m and a functor $F_2 : (Y)_1^m \longrightarrow X_i$;
- (5c) a choice of k categories Z_1, \dots, Z_k and a functor $F_3 : (Z)_1^k \longrightarrow X_j$;
- (5d) a choice of a functor $G_1 : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \longrightarrow X$ and a natural isomorphism $\phi_1 : G_1 \longrightarrow F_1 \circ_i F_2$;
- (5e) a choice of a functor $G_2 : (X)_1^{j-1} \times (Z)_1^k \times (X)_{j+1}^n \longrightarrow X$ and a natural isomorphism $\phi_2 : G_2 \longrightarrow F_2 \circ_j F_3$;
- (5f) a choice of a functor $K : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^{j-1} \times (Z)_1^k \times (X)_{j+1}^n \longrightarrow X$ and two natural isomorphisms

$$\psi_1 : K \longrightarrow F_1 \circ_i G_2, \quad \psi_2 : K \longrightarrow G_1 \circ_{\tilde{j}} F_3$$

where $\tilde{j} = i + j - 1$ (we suppose $j > i$), such that the following diagram of natural isomorphisms is commutative:

$$\begin{array}{ccc} K & \xrightarrow{\psi_1} & F_1 \circ_i G_2 \\ \psi_2 \downarrow & & \downarrow F_1 \circ_i \phi_2 \\ G_1 \circ_i F_3 & \xrightarrow{\phi_1 \circ_{\tilde{j}} F_3} & F_1 \circ_{i,j} (F_2, F_3) \end{array}$$

We are going to illustrate with some examples the dendroidal structure of $wCat^1$ in the context described above. Let $T = \text{Cor}_n \circ_i \text{Cor}_m$, hence a dendrex α of shape T is the same thing as the data described in (3) above. If $\partial : \text{Cor}_n \longrightarrow T$ is the obvious outer face of T then $\partial^*(\alpha)$ corresponds to the choice of the categories X_1, \dots, X_n, X and the functor $F_1 : (X)_1^n \longrightarrow X$. If $\partial : \text{Cor}_{n+m-1} \longrightarrow T$ is the inner face of T then $\partial^*(\alpha)$ corresponds to the choice of the categories $X_1, \dots, X_{i-1}, Y_1, \dots, Y_m, X_{i+1}, \dots, X_n, X$ and the functor $G : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \longrightarrow X$. (The choice of G instead of $F_1 \circ_i F_2$ follows from the definition of the map of operads $\partial : W\Omega(\text{Cor}_{n+m-1}) \longrightarrow W\Omega(T)$.)

One can similarly decipher what a degeneracy looks like. A simple case of such occurs when $R = \text{Cor}_n \circ_i \text{Cor}_1$, $T = \text{Cor}_n$ and $\sigma : R \longrightarrow T$ is the degeneracy in question. If β is a dendrex of shape T , that is a choice of categories X_1, \dots, X_n, X and a functor $F_1 : (X)_1^n \longrightarrow X$, then $\sigma^*(\beta)$ adds to the information contained in β the identity functor $\text{id} : X_i \longrightarrow X_i$, and the identity natural transformation $F_1 \longrightarrow F_1 \circ_i \text{id}$.

4.2 Weak 2-categories

We turn our attention now to the dendroidal set $wCat^2$. Our goal is to unpack the definition and compare the result with bicategories. It will become apparent later that the right notion to compare the data of $wCat^2$ contained in lower degrees is unbiased

bicategories and their homomorphisms. These notions were defined by Tom Leinster in [8]. We recall them in the Appendix.

The Section is organized as follows: First we analyse the sets $wCat^2$, $wCat^2_{\text{Cor}_1}$ and their relations to unbiased bicategories, and we prove that the category of unbiased bicategories is isomorphic to the homotopy category of dendroidal weak 2-categories. Then we conclude the Section by a conjecture that predicts a stronger relation between bicategories and dendroidal weak 2-categories.

4.2.1 Dendroidal weak 2-categories

In this Subsection we analyse those components of the dendroidal set $wCat^2$ which will correspond to bicategories and homomorphisms of bicategories.

4.2.2 Dendrices of shape $|$

Since

$$(wCat^2)_| = \left(\int_{\text{Sets}} Cat(\text{hcN}_d(\mathcal{C}tg))_- \right)_|,$$

the definition of the Grothendieck construction implies that an element of $(wCat^2)_|$ is a pair (A, x) , where A is a set and x is a dendrex of shape $|$ in the dendroidal set $Cat(\text{hcN}_d(\mathcal{C}tg))_A$. Hence

$$x \in dSets(N_d(As_A) \otimes \Omega[|], \text{hcN}_d(\mathcal{C}tg)) = dSets(N_d(As_A), \text{hcN}_d(\mathcal{C}tg)).$$

Since $\text{hcN}_d(\mathcal{C}tg)$ is 3-coskeletal, it is enough to look at the degree 0, 1, 2 and 3 components of x .

- (0) The degree 0 component of x is the map of sets

$$x_| : N_d(As_A)_| \longrightarrow \text{hcN}_d(\mathcal{C}tg)_|.$$

Since $N_d(As_A)_|$ consists of the objects of the operad As_A and $\text{hcN}_d(\mathcal{C}tg)_|$ consists of categories, it follows that $x_|$ is the same thing as the choice of a category $\mathcal{A}(a_1, a_2)$ for each ordered pair $(a_1, a_2) \in A \times A$.

- (1) Let us look at the x_{Cor_n} component, $n \in \mathbb{N}$. There are three cases to distinguish. First, an element in $N_d(As_A)_{\text{Cor}_0}$ consists of a pair (a, a) where $a \in A$, and the operation $* \in As_A(\quad; (a, a))$. We have seen in Subsection 4.1 that an element of $\text{hcN}_d(\mathcal{C}tg)_{\text{Cor}_0}$ is a category together with an object of it. Since x has to be

compatible with the face map $| \longrightarrow \text{Cor}_0$, it follows that x_{Cor_0} picks for each $a \in A$ a functor $\Psi_a : * \longrightarrow \mathcal{A}(a, a)$.

Second, an element in $N_d(As_A)_{\text{Cor}_1}$ consists of a pair $(a_1, a_2) \in A^2$ and the operation $* \in As_A((a_1, a_2); (a_1, a_2))$, which is also the corresponding unit operation in the operad As_A . An element of $\text{hcN}_d(\mathcal{C}tg)_{\text{Cor}_1}$ is a functor between two chosen categories. Again, since x has to be compatible with the various face and degeneracy maps, it follows that x_{Cor_1} amounts to choosing the identity functor on every already chosen category $\mathcal{A}(a_1, a_2)$. Hence x_{Cor_1} does not contribute with any new information.

Third, for $n \geq 2$ x_{Cor_n} picks for each admissible ordered sequence

$$\sigma = ((a_1, a_2), (a_2, a_3), \dots, (a_{n-1}, a_n); (a_1, a_n))$$

of $n + 1$ objects of As_A a functor

$$\Psi_\sigma : \mathcal{A}(a_1, a_2) \times \dots \times \mathcal{A}(a_{n-1}, a_n) \longrightarrow \mathcal{A}(a_1, a_n).$$

We can include the cases $n = 0, 1$ in the third one in the obvious way.

- (2) The degree 2 component of x consists of x_T where $T = \text{Cor}_n \circ_i \text{Cor}_m$ for the various $n, m, i \in \mathbb{N}$, $n \neq 0$. There are face maps into the tree T from the m, n and $m + n - 1$ corollas, and in case $n = 1$ or $m = 1$ there are also degeneracy maps with T as the domain. Since x has to be compatible with these faces and degeneracies, we can conclude that x_T provides the following bit of extra data: For any pair of admissible ordered sequences

$$\begin{aligned} \sigma &= ((a_1, a_2), \dots, (a_{n-1}, a_n); (a_1, a_n)), \\ \rho &= ((a_i, b_2), (b_2, b_3), \dots, (b_{m-1}, a_{i+1}); (a_i, a_{i+1})) \end{aligned}$$

and any $1 \leq i \leq n$ a natural isomorphism

$$\phi_{\sigma, \rho, i} : \Psi_{\sigma \circ_i \rho} \longrightarrow \Psi_\sigma \circ_i \Psi_\rho,$$

There is one condition on these natural isomorphisms: in case $n = 1$ or $m = 1$, the corresponding natural isomorphism has to be the identity (it follows from the compatibility with degeneracies again).

- (3) The degree 3 components of x do not give rise to any extra data, but the dendroidal identities with face maps induce relations on the already existing one, corresponding to cases (4f) and (5f) of Subsection 4.1. Explicitly, the diagrams of functors

$$\begin{array}{ccc} \Psi_{\sigma \circ_i \rho \circ_j \tau} & \xrightarrow{\phi} & \Psi_{\sigma \circ_i \rho} \circ_j \Psi_\tau \\ \phi \downarrow & & \downarrow \phi \circ_j \Psi \\ \Psi_\sigma \circ_i \Psi_{\rho \circ_j \tau} & \xrightarrow{\Psi \circ_i \phi} & \Psi_\sigma \circ_i \Psi_\rho \circ_j \Psi_\tau \end{array}$$

$$\begin{array}{ccc}
\Psi_{\sigma \circ_{i,j}(\rho, \tau)} & \xrightarrow{\phi} & \Psi_{\sigma \circ_i \rho} \circ_j \Psi_{\tau} \\
\downarrow \phi & & \downarrow \phi \circ_j \Psi \\
\Psi_{\sigma \circ_j \tau} \circ_i \Psi_{\rho} & \xrightarrow{\phi \circ_i \Psi} & \Psi_{\sigma} \circ_{i,j}(\Psi_{\rho}, \Psi_{\tau})
\end{array}$$

are commutative.

4.2.3 Dendrices of shape Cor_1

An element of $(w\mathcal{Cat}^2)_{\text{Cor}_1}$ consists of pairs (f, y) where $f : A \longrightarrow B$ is a map of sets and

$$y : \Omega[\text{Cor}_1] \longrightarrow \coprod_{S \in \mathcal{S}ets} \underline{dSets}(N_d(As_S), w\mathcal{Cat}^1)$$

has three relevant components:

$$\begin{aligned}
y_A &\in dSets(N_d(As_A), w\mathcal{Cat}^1), \\
y_B &\in dSets(N_d(As_B), w\mathcal{Cat}^1), \\
y_f &\in dSets(N_d(As_A \otimes \Omega(\text{Cor}_1)), w\mathcal{Cat}^1).
\end{aligned}$$

These three components are related by the compatibility condition of the Grothendieck construction in the following way. Let Cor_1 be represented by the tree

$$\begin{array}{c}
| \\
1 \\
\bullet \\
v \\
0
\end{array}$$

thus the set of colours of the operad $\Omega(\text{Cor}_1)$ is $\{0, 1\}$ and the only non-trivial operation is $v \in \Omega(\text{Cor}_1)(1; 0)$. If $\partial_1, \partial_0 : | \longrightarrow \text{Cor}_1$ denote the face maps sending $|$ to the leaf and root of Cor_1 respectively then $\partial_1^*(y_f) = y_A$ and $\partial_0^*(y_f) = f^*(y_B)$. The components y_A and y_B were described in the first part of Subsection 4.2.2, hence we need to describe y_f only.

Let us recall first the operad $As_A \otimes \Omega(\text{Cor}_1)$ in more detail. The set of colours of this operad contains all pairs (\mathfrak{a}, l) where $\mathfrak{a} = (a_1, a_2) \in A^2$ is a colour of As_A and $l \in \{0, 1\}$ is a colour of $\Omega(\text{Cor}_1)$. The operations are generated by the following three types of basic ones:

$$\begin{array}{c}
(\mathfrak{a}, 1) \\
| \\
(\mathfrak{a}, v) \circ \\
| \\
(\mathfrak{a}, 0)
\end{array}$$

is a picture of the basic operation $(\mathfrak{a}, v) \in \Omega(\text{Cor}_1)((\mathfrak{a}, 1); (\mathfrak{a}, 0))$ induced by $v \in \Omega(\text{Cor}_1)(1; 0)$ for any $\mathfrak{a} = (a_1, a_2) \in A^2$, and

$$(4-1) \quad \begin{array}{c} (\mathfrak{a}_1, 0) \quad (\mathfrak{a}_2, 0) \quad \dots \quad (\mathfrak{a}_n, 0) \\ \diagdown \quad \diagup \quad \quad \quad \diagup \\ \bullet \\ (*) , 0 \\ \mid \\ (\mathfrak{a}, 0) \end{array}$$

$$(4-2) \quad \begin{array}{c} (\mathfrak{a}_1, 1) \quad (\mathfrak{a}_2, 1) \quad \dots \quad (\mathfrak{a}_n, 1) \\ \diagdown \quad \diagup \quad \quad \quad \diagup \\ \bullet \\ (*) , 1 \\ \mid \\ (\mathfrak{a}, 1) \end{array}$$

are pictures of the basic operations in $As_A \otimes \Omega(\text{Cor}_1)((\mathfrak{a}_1, 0), \dots, (\mathfrak{a}_n, 0); (\mathfrak{a}, 0))$ and $As_A \otimes \Omega(\text{Cor}_1)((\mathfrak{a}_1, 1), \dots, (\mathfrak{a}_n, 1); (\mathfrak{a}, 1))$ respectively, induced by the unique operation

$$* \in As_A((a_1, a_2), (a_2, a_3), \dots, (a_n, a_{n+1}); (a_1, a_{n+1}))$$

where $\mathfrak{a}_i = (a_i, a_{i+1}) \in A^2$ and $\mathfrak{a} = (a_1, a_{n+1}) \in A^2$. The operations generated this way are subject to the relations which imply that the obvious projections $As_A \otimes \Omega(\text{Cor}_1) \rightarrow As_A$, $As_A \otimes \Omega(\text{Cor}_1) \rightarrow \Omega(\text{Cor}_1)$ are maps of operads, and to the following relation (interchange law in the Boardman-Vogt tensor product for operads):

$$\begin{array}{c} (\mathfrak{a}_1, 1) \quad (\mathfrak{a}_2, 1) \quad (\mathfrak{a}_n, 1) \\ \mid \quad \mid \quad \mid \\ (\mathfrak{a}_1, v) \quad (\mathfrak{a}_2, v) \quad (\mathfrak{a}_n, v) \\ \diagdown \quad \diagup \quad \quad \quad \diagup \\ \bullet \\ (*) , 0 \\ \mid \\ (\mathfrak{a}, 0) \end{array} = \begin{array}{c} (\mathfrak{a}_1, 1) \quad (\mathfrak{a}_2, 1) \quad (\mathfrak{a}_n, 1) \\ \diagdown \quad \diagup \quad \quad \quad \diagup \\ \bullet \\ (*) , 1 \\ \mid \\ (\mathfrak{a}, 1) \\ \mid \\ (\mathfrak{a}, v) \\ \mid \\ (\mathfrak{a}, 0) \end{array}$$

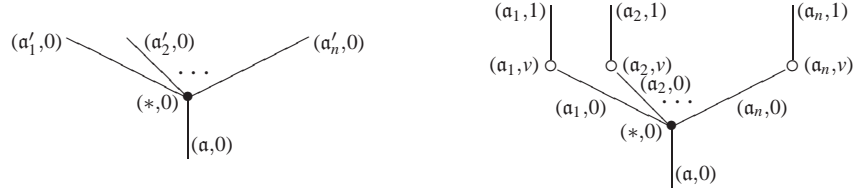
The properties of the operad $As_A \otimes \Omega(\text{Cor}_1)$ imply

Lemma 4.1 *For any ordered sequence $\sigma = ((\mathfrak{a}_1, l_1), \dots, (\mathfrak{a}_n, l_n); (\mathfrak{a}, l))$, the corresponding set of operations $As_A \otimes \Omega(\text{Cor}_1)(\sigma)$ contains at most one element. Moreover, in case $As_A \otimes \Omega(\text{Cor}_1)(\sigma)$ is not empty, $\mathfrak{a}_i = (a_i, a_{i+1})$ and $\mathfrak{a} = (a_1, a_{n+1})$ for some $a_1, \dots, a_{n+1} \in A$.*

Proof When $l = 1$, the only possibilities for the sequence σ that give nonempty $As_A \otimes \Omega(\text{Cor}_1)(\sigma)$ are the ones corresponding to (4-2). Each such set of operations contains exactly one element: $(*, 1)$. If $l = 0$ and all $l_i = 0$, then again the only nonempty sets of operations are the ones corresponding to (4-1), with a unique operation $(*, 0)$ in each.

The remaining cases to study are the ones when σ is such that $l = 0$ and there exists i with $l_i = 1$. Suppose that in such a case $As_A \otimes \Omega(\text{Cor}_1)(\sigma)$ is not empty. The

interchange law implies then that these operations can always be reduced to a form, which can be visualized by a tree resulting from iterated gluing of the following two types of operations:

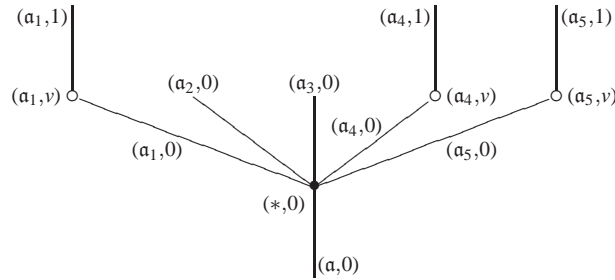


where “gluing” means identifying the two roots indexed by $(a, 0)$. We can conclude that $a_i = (a_i, a_{i+1})$, $a = (a_1, a_{n+1})$ for some $a_1, \dots, a_{n+1} \in A$ and $As_A \otimes \Omega(\text{Cor}_1)(\sigma)$ contains again exactly one element.

Let us illustrate the latter situation in a particular example, for instance when

$$\sigma = ((a_1, 1), (a_2, 0), (a_3, 0), (a_4, 1), (a_5, 1); (a, 0)).$$

Any operation in $As_A \otimes \Omega(\text{Cor}_1)(\sigma)$ can be reduced to the form



Since the set of operations is assumed to be nonempty, $a_i = (a_i, a_{i+1})$ for all $1 \leq i \leq 6$ and $a = (a_1, a_6)$, and there is exactly one element in the set $As_A \otimes \Omega(\text{Cor}_1)(\sigma)$ (that is the composite illustrated by the tree above). \square

We can use this discussion about the operad $As_A \otimes \Omega(\text{Cor}_1)$ to understand the y_f component of an element $y \in w\mathcal{Cat}_{\text{Cor}_1}^2$. The degree 0 component of y_f consists of the map of sets

$$(y_f)_| : N_d(As_A \otimes \Omega(\text{Cor}_1))_| \longrightarrow (w\mathcal{Cat}^1)_|,$$

thus $(y_f)_|$ amounts to choosing categories $\mathcal{C}_1(a_1, a_2)$ and $\mathcal{C}_0(a_1, a_2)$ for every pair $(a_1, a_2) \in A^2$. The compatibility condition in the Grothendieck construction implies that these categories are the corresponding ones appearing in the definitions of y_A and y_B . Explicitly,

$$\begin{aligned} \mathcal{C}_1(a_1, a_2) &= \mathcal{A}(a_1, a_2) \text{ of } (y_A)_|, \\ \mathcal{C}_0(a_1, a_2) &= \mathcal{B}(f(a_1), f(a_2)) \text{ of } (y_B)_|. \end{aligned}$$

Lemma 4.1 implies that y_f in degree 1 amounts to the choice of a functor

$$\Psi_\sigma: \mathcal{C}_{l_1}(a_1, a_2) \times \cdots \times \mathcal{C}_{l_{n-1}}(a_{n-1}, a_n) \longrightarrow \mathcal{C}_l(a_1, a_n)$$

for every ordered sequence

$$\sigma = (((a_1, a_2), l_1), ((a_2, a_3), l_2), \dots, ((a_{n-1}, a_n), l_{n-1}); ((a_1, a_n), l))$$

of objects of $As_A \otimes \Omega(\text{Cor}_1)$. A particular case of such a functor is

$$\Psi_\sigma: \mathcal{A}(a_1, a_2) \longrightarrow \mathcal{B}(f(a_1), f(a_2)).$$

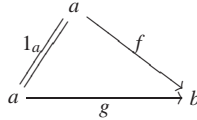
As a consequence of the compatibility conditions of the Grothendieck construction, in case $l = l_i = 1$ or $l = l_i = 0$ for all i , one gets back the corresponding functors Ψ^A or Ψ^B respectively, resulting from y_A or y_B (we described them in Subsection 4.2.2).

The data contained in degree 2 of y_f amounts to choices of natural transformations $\phi: \Psi_\sigma \circ \Psi_\rho \longrightarrow \Psi_{\sigma \circ \rho}$, and similarly, the information contained in degree 3 can be described with the same semantics as in the description of $wCat_1^2$.

4.2.4 The relation between bicategories and dendroidal weak 2-categories

In this Subsection we aim to establish a relation between dendroidal weak 2-categories and (unbiased) bicategories. Let us recall first that the homotopy category of an inner Kan complex X in simplicial sets is the category $ho(X) := \tau(X)$, where τ is the left adjoint of the nerve functor $N: Cat \longrightarrow sSets$. The category $ho(X)$ is defined as follows:

- the objects of $ho(X)$ are the elements of X_0 ;
- the arrows of $ho(X)$ are equivalence classes of elements of X_1 , where the equivalence relation is left homotopy: if $a, b \in X_0$ and $f, g: a \longrightarrow b$ are elements of $X(a, b) \subseteq X_1$, we say that f is left homotopic to g if there exists an element of X_2 that fills the triangle



Remark 4.2 We could have chosen to define the equivalence relation above with the dual notion of right homotopy, since X being an inner Kan complex implies that these two notions are the same. In fact, this is the only reason we renamed the category $\tau(X)$: the description of this category is much easier when X is an inner Kan complex.

The restriction functor $i^*: dSets \longrightarrow sSets$ preserves inner Kan complexes, hence it makes sense to talk about the category $ho(i^*(wCat^2))$ (we will call it *the category of dendroidal weak 2-categories*). Our goal is to compare this category with $ubiCtg$.

The description of the elements of $wCat_1^2$ in Subsection 4.2.2 can be summarized in

Proposition 4.3 *The objects of the category $ho(i^*(wCat^2))$ are in one-to-one correspondence with the objects of $ubiCtg$.*

Let us turn to the comparison of morphisms of the categories in question. By Proposition 4.3 we have defined a functor $\Phi: ubiCtg \longrightarrow ho(i^*(wCat^2))$ on the objects. We complete the definition of Φ by

Proposition 4.4 *For any $\mathbb{A}, \mathbb{B} \in ubiCtg$ there is a one-to-one correspondence between the hom-sets $ubiCtg(\mathbb{A}, \mathbb{B})$ and $ho(i^*(wCat^2))(\Phi(\mathbb{A}), \Phi(\mathbb{B}))$. This correspondence is functorial.*

Proof Suppose that $(F, f): \mathbb{A} \longrightarrow \mathbb{B}$ is a map of unbiased bicategories (with strict unit). With the use of the description of $wCat_{Cor_1}^2$ we gave in Subsection 4.2.2 first we define an $y^F \in wCat_{Cor_1}^2$, induced by (F, f) . Recall, that such an y^F is determined by three components, and the components y_A^F, y_B^F are obvious.

Let us define the component $y_f^F: N_d(As_A) \otimes \Omega[Cor_1] \longrightarrow wCat^1$. In degree 0 again the definition of y_f^F is obvious, the first non-trivial choices we have to make arise in degree 1. We need to give the components $(y_f^F)_{Cor_n}$ for every $n \geq 1$, i.e. a functor

$$\Psi_\sigma: \mathcal{C}_{l_1}(a_1, a_2) \times \mathcal{C}_{l_2}(a_2, a_3) \times \cdots \times \mathcal{C}_{l_{n-1}}(a_{n-1}, a_n) \longrightarrow \mathcal{B}(f(a_1), f(a_n))$$

for every ordered sequence σ with $a_i \in A$, $l_i \in \{0, 1\}$, where

$$\mathcal{C}_{l_i}(a_i, a_{i+1}) = \begin{cases} \mathcal{A}(a_i, a_{i+1}) & \text{if } l_i = 1, \\ \mathcal{B}(f(a_i), f(a_{i+1})) & \text{if } l_i = 0. \end{cases}$$

Define this functor to be the composite

$$\begin{array}{ccc} \mathcal{C}_{l_1}(a_1, a_2) \times \cdots \times \mathcal{C}_{l_{n-1}}(a_{n-1}, a_n) & & \\ \downarrow G & \searrow & \\ \mathcal{B}(f(a_1), f(a_2)) \times \cdots \times \mathcal{B}(f(a_{n-1}), f(a_n)) & \xrightarrow{\Psi_\sigma^\mathcal{B}} & \mathcal{B}(f(a_1), f(a_n)) \end{array}$$

where G is the product of the functors $\mathcal{C}_{l_i}(a_i, a_{i+1}) \longrightarrow \mathcal{B}(f(a_i), f(a_{i+1}))$ that are either $F_{(a_i, a_{i+1})}$ or identity, depending on l_i . These choices define y_f^F in degree 1.

The next step is to give the components of y_f^F in degree 2, that is we have to give natural isomorphisms $\Psi_\sigma \circ \Psi_\rho \longrightarrow \Psi_{\sigma \circ \rho}$. It is a straightforward computation to see that for any such natural isomorphism we have exactly one choice: some pasting of the invertible 2-cells in the data defining the map of bicategories (F, f) , and any such pasting is unique due to the coherence conditions on (F, f) . As a consequence, the relations imposed in degree 3 for y_f^F are also satisfied, hence y^F is indeed an element of $wCat^2(\Phi(\mathbb{A}), \Phi(\mathbb{B})) \subseteq wCat_{\text{Cor}_1}^2$.

This far we have constructed a map of sets Φ :

$$ubiCtg(\mathbb{A}, \mathbb{B}) \longrightarrow wCat^2(\Phi(\mathbb{A}), \Phi(\mathbb{B})) \longrightarrow ho(i^*(wCat^2))(\Phi(\mathbb{A}), \Phi(\mathbb{B}))$$

that is clearly functorial. We still need to show that Φ is surjective and injective.

To treat surjectivity, for any $y \in wCat^2(\Psi(\mathbb{A}), \Psi(\mathbb{B})) \subseteq wCat_{\text{Cor}_1}^2$ we construct a homomorphism of unbiased bicategories $(F, f): \mathbb{A} \longrightarrow \mathbb{B}$ such that the associated y^F described above will be in the class of y in the homotopy category. For any such y it is straightforward how to get the map of sets $f: A \longrightarrow B$ and the functors $F_{(a_1, a_2)}: \mathcal{A}(a_1, a_2) \longrightarrow \mathcal{B}(f(a_1), f(a_2))$, hence we only need to construct the natural isomorphisms for the data defining (F, f) , displayed in diagram (A–1). We will discuss only the case $n = 2$, the general case can be treated analogously. These natural isomorphisms are obtained as the composite of two natural isomorphisms given by the degree 2- and 3 data in y_f :

- (a) The first natural isomorphism is the one in in $(y_f)_{\text{Cor}_1 \circ \text{Cor}_2}$:

$$\begin{array}{ccc} \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) & \xrightarrow{\Psi^A} & \mathcal{A}(a_1, a_3) \\ & \searrow K & \downarrow F_{13} \\ & & \mathcal{B}(f(a_1), f(a_2)) \end{array}$$

$\uparrow \alpha$

- (b) The second natural isomorphism comes from a pasting diagram, induced by the

data in $(y_f)_{\text{Cor}_2 \circ (\text{Cor}_1, \text{Cor}_1)}$:

$$\begin{array}{ccccc}
 & & & \mathcal{B} \times \mathcal{B} & \\
 & & \text{id} \times F_{23} \nearrow & \downarrow \Psi^{\mathcal{B}} & \\
 & \mathcal{B}(f(a_1), f(a_2)) \times \mathcal{A}(a_2, a_3) & \uparrow \alpha_2 & & \\
 F_{12} \times \text{id} \nearrow & & \uparrow \beta_1 & \text{---} H \text{---} & \\
 \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) & \xrightarrow{K} & \mathcal{B}(f(a_1), f(a_3)) & & \\
 \text{id} \times F_{23} \searrow & & \downarrow \beta_2 & \text{---} G \text{---} & \\
 & \mathcal{A}(a_1, a_2) \times \mathcal{B}(f(a_2), f(a_3)) & \downarrow \alpha_1 & & \\
 & \searrow F_{12} \times \text{id} & & \downarrow \Psi^{\mathcal{B}} & \\
 & & & \mathcal{B} \times \mathcal{B} &
 \end{array}$$

Note that the coherence conditions in $(y_f)_{\text{Cor}_2 \circ (\text{Cor}_1, \text{Cor}_1)}$ imply $\alpha_2 \cdot \beta_1 = \alpha_1 \cdot \beta_2$ as natural isomorphisms $K \Rightarrow \Psi^{\mathcal{B}} \circ (F_{12} \times F_{23})$.

We can set the required natural isomorphism for the data in the homomorphism (F, f) to be $\alpha \circ (\alpha_2 \cdot \beta_1)$. The coherence conditions in the degree 3 components of y_f imply that $(F, f): \mathbb{A} \longrightarrow \mathbb{B}$ is indeed a homomorphism of unbiased bicategories (with strict unit). The associated $y^F \in w\text{Cat}^2(\Phi(\mathbb{A}), \Phi(\mathbb{B}))$ is homotopic to y since we can construct an element in $w\text{Cat}_{\text{Cor}_1 \circ \text{Cor}_1}^2$ with faces $\text{id}_{\Phi(\mathbb{A})}$, y and y^F . Hence the function $\Phi: \text{ubiCtg}(\mathbb{A}, \mathbb{B}) \longrightarrow \text{ho}(i^* w\text{Cat}^2)(\Phi(\mathbb{A}), \Phi(\mathbb{B}))$ is surjective as well. This construction shows that Φ is injective as well and the proof is finished. \square

Propositions 4.3 and 4.4 imply immediately

Theorem 4.5 *The categories ubiCtg and $\text{ho}(i^*(w\text{Cat}^2))$ are isomorphic. Hence the category of classical bicategories is equivalent to the category of dendroidal weak 2-categories.*

To conclude this Section, we conjecture that the following, stronger statement is true:

Conjecture 4.6 *The inclusion of simplicial sets $N(\text{ubiCtg}) \longrightarrow i^*(w\text{Cat}^2)$ is a weak equivalence in the Joyal model structure on $s\text{Sets}$.*

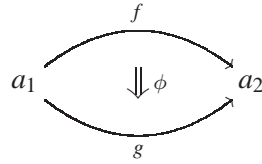
A Notions of bicategories

The notion of bicategory first appeared explicitly in the paper of Bénabou [1]. Intuitively, bicategories are generalised categories where the composition of arrows is not strictly associative, only up to some coherent 2-cells which are part of the structure. The theory of bicategories had a quick development, due to the usefulness of the notion in different fashionable areas of mathematics. Amongst these areas we can find ordinary category theory: as Ross Street states in [13], many fundamental constructions of categories are bicategorical in nature. Bicategories can be considered as generalisations of monoidal categories as well, giving new insight to the theory of monoidal categories. Another area where bicategories were influential is algebraic topology, especially higher homotopy theory: bicategories are the first step in the build-up of higher categories and groupoids, which should provide algebraic models of homotopy n -types.

A.1 Classical bicategories

A bicategory \mathbb{A} consists of the following data and axioms:

- (D1) a set A , called the set of objects or 0-cells;
- (D2) for every ordered pair of objects $(a_1, a_2) \in A \times A$ a category $\mathcal{A}(a_1, a_2)$. The objects of such a category are called arrows or 1-cells of \mathbb{A} , the maps are called 2-cells of \mathbb{A} . If $f, g \in \mathcal{A}(a_1, a_2)$ are 1-cells and $\phi \in \mathcal{A}(a_1, a_2)(f, g)$ is a 2-cell between them, we usually depict this situation as $f \xRightarrow{\phi} g$ or as



The composition of 2-cells in a category $\mathcal{A}(a_1, a_2)$ is called vertical composition and for two composable 2-cells ϕ, ϕ' the composite is denoted by juxtaposition: $\phi\phi'$.

- (D3) functors which define horizontal composition and units in \mathbb{A} :

- (D3a) for all $(a_1, a_2, a_3) \in A^3$, $\psi: \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) \longrightarrow \mathcal{A}(a_1, a_3)$. We denote by “ \cdot ” the horizontal composite of two 1-cells (and two 2-cells), thus $g \cdot f := \Psi(f, g)$ etc.
- (D3b) for all $a \in A$, $\psi_0: * \longrightarrow \mathcal{A}(a, a)$, that is a 1-cell $\text{Id}_a \in \mathcal{A}(a, a)$.

- (D4a) natural isomorphisms, relating the two different ways of horizontal compositions of three 1-cells in \mathbb{A} : for all $a_1, a_2, a_3, a_4 \in A$

$$\begin{array}{ccc}
 \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) \times \mathcal{A}(a_3, a_4) & \xrightarrow{\text{id} \times \psi} & \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_4) \\
 \downarrow \psi \times \text{id} & \uparrow \alpha & \downarrow \psi \\
 \mathcal{A}(a_1, a_3) \times \mathcal{A}(a_3, a_4) & \xrightarrow{\psi} & \mathcal{A}(a_1, a_4)
 \end{array}$$

that is, invertible 2-cells $(h \cdot g) \cdot f \xRightarrow{\alpha} h \cdot (g \cdot f)$ in \mathbb{A} for any three composable 1-cells f, g, h .

- (D4b) natural isomorphisms, relating composition with units to the identity: for all $a_1, a_2 \in A$,

$$\begin{array}{ccc}
 \mathcal{A}(a_1, a_2) \times * & \xrightarrow{\text{id} \times \psi_0} & \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_2) \\
 \searrow & \lambda \swarrow & \downarrow \psi \\
 & & \mathcal{A}(a_1, a_2) \\
 * \times \mathcal{A}(a_1, a_2) & \xrightarrow{\psi_0 \times \text{id}} & \mathcal{A}(a_1, a_1) \times \mathcal{A}(a_1, a_2) \\
 \searrow & \rho \swarrow & \downarrow \psi \\
 & & \mathcal{A}(a_1, a_2)
 \end{array}$$

that is, for any 1-cell $f \in \mathcal{A}(a_1, a_2)$ invertible 2-cells

$$\text{Id}_{a_2} \cdot f \xRightarrow{\lambda} f \quad \text{and} \quad f \cdot \text{Id}_{a_1} \xRightarrow{\rho} f.$$

The data given above is subject to two axioms that ensure that the various associativity and unit constraints α, ρ, λ compose coherently:

- (A1) The following pentagon commutes for any involved composable 1-cells

$$\begin{array}{ccccc}
 & & ((k \cdot h) \cdot g) \cdot f & \xRightarrow{\alpha \cdot \text{id}_f} & (k \cdot (h \cdot g)) \cdot f \\
 & \swarrow \alpha & & & \searrow \alpha \\
 (k \cdot h) \cdot (g \cdot f) & & & & k \cdot ((h \cdot g) \cdot f) \\
 & \searrow \alpha & & \swarrow \text{id}_k \cdot \alpha & \\
 & & k \cdot (h \cdot (g \cdot f)) & &
 \end{array}$$

(A2) The following triangle commutes for any involved composable 1-cells

$$\begin{array}{ccc}
 (g \cdot \text{Id}) \cdot f & \xrightarrow{\alpha} & g \cdot (\text{Id} \cdot f) \\
 \searrow \rho \cdot \text{id}_f & & \swarrow \text{id}_g \cdot \lambda \\
 & g \cdot f &
 \end{array}$$

Example A.1 Any (strict) 2-category is a bicategory where the associativity and unit 2-cells α, ρ, λ are all identities.

Example A.2 Any monoidal category \mathcal{C} is a bicategory with one 0-cell. The 1-cells of this bicategory are the objects of \mathcal{C} and the 2-cells are the arrows of \mathcal{C} . The other data and axioms of the bicategory are induced by the monoidal structure on \mathcal{C} , in the obvious way.

Example A.3 There exists a bicategory $\mathbb{B}\text{Mod}$, defined as follows:

- The 0-cells of $\mathbb{B}\text{Mod}$ are rings with unit A, B, \dots
- The category of (A, B) -bimodules defines the 1- and 2-cells of $\mathbb{B}\text{Mod}$.
- Horizontal composition, units, etc. are given by tensor product of bimodules.

A.2 Homomorphisms of classical bicategories

There exist a number of notions of homomorphisms of bicategories, the one we define here is that of weak homomorphisms in the literature. Thus, for us a homomorphism of bicategories $(F, f): \mathbb{A} \longrightarrow \mathbb{B}$ consists of the following data and axioms:

(D1) A function $f: A \longrightarrow B$ from the set of 0-cells of \mathbb{A} to the set of 0-cells of \mathbb{B} .

(D2) For every ordered pair of 0-cells $(a_1, a_2) \in A^2$ a functor

$$F_{a_1 a_2}: \mathcal{A}(a_1, a_2) \longrightarrow \mathcal{B}(f(a_1), f(a_2)).$$

(D3a) For every $(a_1, a_2, a_3) \in A^3$ natural isomorphisms relating horizontal compositions and F :

$$\begin{array}{ccc}
 \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) & \xrightarrow{\psi^{\mathbb{A}}} & \mathcal{A}(a_1, a_3) \\
 \downarrow F \times F & \uparrow \theta & \downarrow F \\
 \mathcal{B}(f(a_1), f(a_2)) \times \mathcal{B}(f(a_2), f(a_3)) & \xrightarrow{\psi^{\mathbb{B}}} & \mathcal{B}(f(a_1), f(a_3))
 \end{array}$$

that is, invertible 2-cells $F(h) \cdot F(g) \xRightarrow{\theta} F(h \cdot g)$ for any composable 1-cells $h, g \in \mathbb{A}$.

(D3b) For every $a \in A$ natural isomorphisms relating units and F :

$$\begin{array}{ccc} * & \xrightarrow{\psi_0^{\mathbb{A}}} & \mathcal{A}(a, a) \\ \parallel & \uparrow \theta_0 & \downarrow F \\ * & \xrightarrow{\psi_0^{\mathbb{B}}} & \mathcal{B}(f(a), f(a)) \end{array}$$

that is, invertible 2-cells $\text{Id}_{f(a)}^{\mathbb{B}} \xRightarrow{\theta_0} F(\text{Id}_a^{\mathbb{A}})$ for any $a \in A$.

The data described above is subject to axioms that ensure that F is coherent with the various associativity- and unit-constraints:

(A1) For every composable 1-cells $k, h, g \in \mathbb{A}$, the following hexagon of invertible 2-cells commutes

$$\begin{array}{ccccc} & & F(k \cdot h) \cdot Fg & & \\ & \nearrow \theta \cdot \text{id} & & \searrow \theta & \\ (Fk \cdot Fh) \cdot Fg & & & & F((k \cdot h) \cdot g) \\ \alpha^{\mathbb{B}} \Downarrow & & & & \Downarrow F\alpha^{\mathbb{A}} \\ Fk \cdot (Fh \cdot Fg) & & & & F(k \cdot (h \cdot g)) \\ & \searrow \text{id} \cdot \theta & & \nearrow \theta & \\ & & Fk \cdot F(h \cdot g) & & \end{array}$$

(A2) For any 1-cell $g \in \mathcal{A}(a, a')$ the following diagrams of invertible 2-cells commute:

$$\begin{array}{ccc} Fg \cdot \text{Id}_{fa}^{\mathbb{B}} & \xRightarrow{\text{id} \cdot \theta_0} & Fg \cdot F(\text{Id}_a^{\mathbb{A}}) \\ \rho^{\mathbb{B}} \Downarrow & & \Downarrow \theta \\ Fg & \xleftarrow{F(\rho^{\mathbb{A}})} & F(g \cdot \text{Id}_a^{\mathbb{A}}) \end{array} \quad \begin{array}{ccc} \text{Id}_{fa'}^{\mathbb{B}} \cdot Fg & \xRightarrow{\theta_0 \cdot \text{id}} & F(\text{Id}_{a'}^{\mathbb{A}}) \cdot Fg \\ \lambda^{\mathbb{B}} \Downarrow & & \Downarrow \theta \\ Fg & \xleftarrow{F(\lambda^{\mathbb{A}})} & F(\text{Id}_{a'}^{\mathbb{A}} \cdot g) \end{array}$$

Classical bicategories and their homomorphisms form a category that we denote by $biCtg$.

A.3 Unbiased bicategories

As we mentioned in the introductory part of Subsection A, it is more natural to compare dendroidal bicategories (the lower degree terms of the dendroidal set $wCat^2$) with the category of unbiased bicategories and their homomorphisms, notions that were defined

by Tom Leinster in [8]. We will briefly discuss them here, the resulting category of unbiased bicategories will be denoted by \overline{ubiCtg} . Since the categories \overline{ubiCtg} and $biCtg$ are equivalent, it is justified to compare unbiased bicategories instead of the classical ones with dendroidal bicategories.

The idea of unbiased bicategories comes from the observation that the definition of bicategories is “biased” towards a binary horizontal composition of 1-cells and a chosen associator between the two different ways to compose horizontally three 1-cells. One can eliminate this bias by considering a definition which resembles operads, as follows:

- (a) for every $n \in \mathbb{N}$ give a horizontal composition of (composable) n -tuples of 1-cells;
- (b) relate the n -ary compositions for various n -s by some given 2-cells (the associators);
- (c) the associators should be coherent, thus they have to satisfy some obvious relations;
- (d) take care of the unit 1-cells.

When one tries to work out the details of the points given above, one notices that step (b) can be fulfilled in two ways, depending on the preferred “operadic” approach one takes: the \circ_i -approach or the general γ -approach. These definitions are equivalent (the two resulting categories of unbiased bicategories are isomorphic). Leinster in his definition takes the second approach, we will take here the first one:

An unbiased bicategory \mathbb{A} consists of the following data:

- (D1) a set A ;
- (D2) for every $(a_1, a_2) \in A^2$ a category $\mathcal{A}(a_1, a_2)$;
- (D3) for every integer $n \geq 0$ and every sequence of objects $(a_1, a_2, \dots, a_{n+1}) \in A^{n+1}$ an associated functor of n -ary composition

$$\mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) \times \dots \times \mathcal{A}(a_n, a_{n+1}) \xrightarrow{\Psi} \mathcal{A}(a_1, a_{n+1}),$$

we usually denote the n -fold horizontal composition of 1-cells by

$$(g_1 \cdot g_2 \cdot \dots \cdot g_n) := \Psi(g_1, \dots, g_n);$$

- (D4) for all $n, m, i \in \mathbb{N}$ such that $1 \leq i \leq n$, $n \neq 0$ and composable sequences of 1-cells

$$(h_1, h_2, \dots, h_n) \text{ and } (g_1, g_2, \dots, g_m)$$

such that $(g_1 \cdot g_2 \cdot \dots \cdot g_m) = h_i$, natural invertible 2-cells

$$(h_1 \cdot h_2 \cdot \dots \cdot h_n) \xRightarrow{\phi} (h_1 \cdot \dots \cdot h_{i-1} \cdot g_1 \cdot g_2 \cdot \dots \cdot g_m \cdot h_{i+1} \cdot \dots \cdot h_n);$$

(D5) for every 1-cell g an invertible 2-cell $g \xRightarrow{\iota} (g)$.

This data has to satisfy some obvious axioms, ensuring coherence of compositions and units.

A.4 Homomorphisms of unbiased bicategories

Suppose that \mathbb{A} and \mathbb{B} are unbiased bicategories. A homomorphism $\mathbb{A} \xrightarrow{(F,f)} \mathbb{B}$ of unbiased bicategories consists of the following data and axioms:

(D1) a function $f: A \rightarrow B$ between the 0-cells of \mathbb{A} and \mathbb{B} ;

(D2) for every ordered pair of 0-cells $(a_1, a_2) \in A^2$ a functor

$$F_{a_1 a_2}: \mathcal{A}(a_1, a_2) \rightarrow \mathcal{B}(f(a_1), f(a_2));$$

(D3) for every $n \in \mathbb{N}$ and every ordered sequence of 0-cells of \mathbb{A} , $(a_1, \dots, a_n) \in A^n$ natural isomorphisms

$$(A-1) \quad \begin{array}{ccc} \mathcal{A}(a_1, a_2) \times \cdots \times \mathcal{A}(a_{n-1}, a_n) & \xrightarrow{\psi^{\mathbb{A}}} & \mathcal{A}(a_1, a_n) \\ \downarrow F^n & \uparrow \theta & \downarrow F \\ \mathcal{B}(f(a_1), f(a_2)) \times \cdots \times \mathcal{B}(f(a_{n-1}), f(a_n)) & \xrightarrow{\psi^{\mathbb{B}}} & \mathcal{B}(f(a_1), f(a_n)) \end{array}$$

This data again is subject to some coherence axioms, ensuring the compatibility of F with the various associativity- and unit constraints of the involved bicategories.

Remark A.4 The category of unbiased bicategories defined above is denoted by \overline{ubiCtg} . It has a full subcategory $ubiCtg$, whose objects are those unbiased bicategories for which the unit 2-cells of (D5) are all identities. We call them unbiased bicategories with strict unit, but note that this terminology is misleading since there is a chain of fully faithful embeddings of equivalent categories

$$biCtg \subseteq ubiCtg \subseteq \overline{ubiCtg}.$$

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